

On Fitting the Gielis Superformula to Empirical Data

SK Mishra
Dept. of Economics
NEHU, Shillong (India)

Introduction: Nature presents abundant examples of closed curves that may be considered as some sort of deformed ellipses. Gielis (2003) invented a superformula to parameterize such shapes.

The classical formula of ellipse is given as:

$$r(\theta) = \left[\left(\frac{\cos(\theta)}{a} \right)^2 + \left(\frac{\sin(\theta)}{b} \right)^2 \right]^{(-1/2)} ; a, b > 0 \quad \dots (1)$$

that may, without any loss to generality, be expressed as:

$$r(\theta) = \left[\left| \frac{\cos(\frac{4\theta}{4})}{a} \right|^2 + \left| \frac{\sin(\frac{4\theta}{4})}{b} \right|^2 \right]^{(-1/2)} ; a, b > 0 \quad \dots (2)$$

or

$$r(\theta) = \left[\left| \frac{\cos(\frac{m\theta}{4})}{a} \right|^{n_2} + \left| \frac{\sin(\frac{m\theta}{4})}{b} \right|^{n_3} \right]^{(-1/n_1)} ; a, b > 0; m = 4; n_1 = n_2 = n_3 = 2 \quad \dots (3)$$

Thus, we have the formula

$$r(\theta) = f(\theta) \cdot \left[\left| \frac{\cos(\frac{m\theta}{4})}{a} \right|^{n_2} + \left| \frac{\sin(\frac{m\theta}{4})}{b} \right|^{n_3} \right]^{(-1/n_1)} ; a, b, m > 0 \quad \dots (4)$$

Gielis found that almost any (closed) curve can be described in terms of $f(\theta)$, which is any defined function of θ , and the parameters $(a, b, m, n_{j=1,2,3})$ in (4) and called it a superformula.

Let us look at the formulas (1) to (4) above a little differently. The parameters a and b relate to maximal length (along x axis) and width (along y axis). When x and y are divided by a and b respectively, the length and the width are unitized such that all the

points lie on or inside the unit circle. If the parameters satisfy the relationship $(n_1 = n_2 = n_3) \geq 1$, we may view (4) above as some sort of ‘ball’ with a well-defined Minkowski distance (of a point from the origin). The Minkowski distance is measured as

$$d = \left[|u|^p + |v|^p \right]^{1/p}; p \geq 1 \quad \dots (5)$$

Note that this distance satisfies the triangle inequality condition for any real $p \geq 1$, (integer or non-integer). For u and v within the unit ball, p larger than unity would deflate them and $p < 1$ would inflate them (violating the Δ inequality). The exponent $(1/p)$ on the square bracket operating on the sum of rescaled u and v (within the square bracket) would have the role opposite to p . However, in the formula

$$d = \left[|u|^p + |v|^q \right]^{1/w}; (p \neq q \neq w) \geq 1 \quad \dots (6)$$

scaling of u and v within the square bracket are not uniform, nor the rescaling of their sum by the exponent on the square bracket has any particular relation with them. They stretch and squeeze the ball in different proportions in different directions.

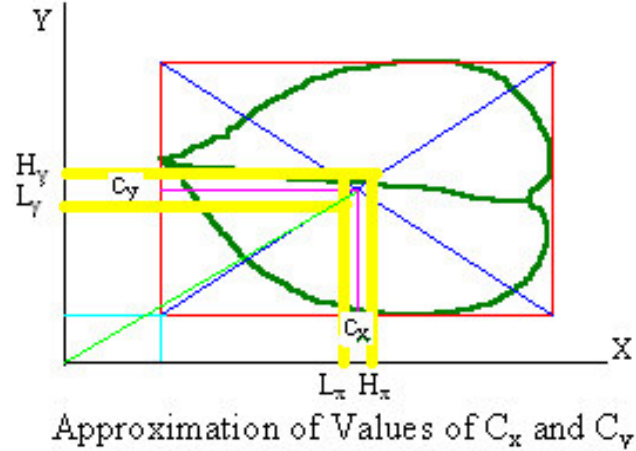
Thus, the matters are entirely different when p, q or w assume a negative or a fractional value. Then d is no longer a measure of distance in the sense described above. Gielis’ formula allows for such negative or fractional values of p, q or w .

Examples in Nature: Among the plant leaves we find different shapes - linear, lanceolate, elliptic, oblong, ovate, abovate, orbicular, cordate, reniform, sagittate, hastate, auricled, etc. Leaves exhibit different patterns in their margin – entire, undulate, dentate, serrate, crenate, pinnately lobed and so on. Leaf tips may be acuminate, acute, obtuse, truncate, retuse, cuspidate, mucronate, and many others. These particulars characterize plant leaves and the plants in turn. Very often, therefore, plant leaves are qualitatively described. However, Gielis holds that it is possible to describe these shapes in terms of parameters $(a, b, m, n_{j=1,2,3})$.

Measurements: A plant leaf may be placed on a graph paper and traced to give a 2d graph, which may be converted into data points $z_i = (x_i, y_i); i = 1, 2, \dots, n$. In practice, these measurements are such that the entire leaf is traced in 4 quadrants, with the centroid placed (approximately) on the origin $(0, 0)$ of the graph. Since it is almost impossible in practice to exactly place the centroid of a leaf on the correct origin, $(0, 0)$, on a graph paper, we assume that the origin is wrongly located at (c_x, c_y) . Accordingly, every point on the trace of the leaf is the distance, r' , measured with the origin (c_x, c_y) on the graph, with the coordinates $(x', y') = (x + c_x, y + c_y)$. After obtaining such data, what remains is to fit the Gielis superformula in the data and obtain the estimates of parameters $(\hat{a}, \hat{b}, \hat{m}, \hat{n}_{j=1,2,3})$. This paper is an attempt to explore this possibility.

The First Step to Estimation of Gielis Parameters: The first and most important task in estimation of the Gielis parameters from such empirical data (with displaced origin) is to

estimate the shift parameters, (c_x, c_y) , so that $(x, y) = (x - c_x, y - c_y)$ can be obtained as correctly as possible. We draw the practically smallest possible rectangle such that the trace of the leaf is inscribed within the said rectangle, we may obtain the first estimate of (c_x, c_y) along with the limits within which each shift parameter would lie. That is: $L_x \leq c_x \leq H_x$; $L_y \leq c_y \leq H_y$. We can also obtain approximate values of a = half of the length and b = half of the width of the rectangle, and the limits within which they lie such that $L_a \leq a \leq H_a$; $L_b \leq b \leq H_b$. Any other method may also be used to guess them.



Having obtained the approximate values of the first four parameters, (c_x, c_y) and (a, b) , searching for the values of the rest four parameters (m, n_1, n_2, n_3) needs a guess work and a non-linear search algorithm.

Empirical Uniqueness of the Parameters of a Formula: Let us define empirical uniqueness of a formula. Given data (v, D) , if a formula $v = \psi(D|\beta)$ obtains tolerably same values of estimates of $(\hat{s}, \hat{\beta})$ for different trials of estimation such that we have $(\hat{s}, \hat{\beta})_1 \cong (\hat{s}, \hat{\beta})_2 \cong (\hat{s}, \hat{\beta})_3 \cong \dots \cong (\hat{s}, \hat{\beta})_k$ for trials 1, 2, 3, ..., k, then the formula $v = \psi(D|\beta)$ is empirically unique in its parameters, β , yielding approximately same measure of distance¹, say, $\hat{s} = \left[\sum [abs\{v - \psi(D|\hat{\beta})\}]^p \right]^{1/p}$, between the observed values (v) and the estimated values, $\hat{v} = \psi(D|\hat{\beta})$. In other words, when the mapping from data space to the parameter space is one-to-many or the mapping from the parameter space to data space is many-to-one the parameters are no longer unique to the set of given data.

The issue of independence among the parameters (nil or negligible possibility of trade-off among them) is important. Parameters have a substantial meaning; they are not simple numbers with inconsequential 'physical' meaning. For example: b in $y = a + bx$ is the velocity, or the rate of change in y as a response to change in x ; a and b in $(x/a)^2 + (y/b)^2 = 1$ relate to the major and the minor of an ellipse. In his superformula Gielis identifies the parameter m with the number of vertices (sub-dimensionality) of a

polygonal figure. The parameter n_1 determines sharpness and flatness of corners and convexity of sides. The parameters n_2 and n_3 determine whether the figure is inscribed ($n_2, n_3 < 2$ i.e. sub-polygon), or circumscribed ($n_2, n_3 > 2$ i.e. super-polygon) in the unit circle (Gielis, 2003, pp. 234-235). However, if the parameters have a strong tendency to trade-off among themselves, their interpretation becomes blurred, if not fully obscure or misleading.

Behaviour of the Estimated Gielis Parameters: We may examine the empirical uniqueness of Gielis superformula. We randomly generate 100 points with the parameters given in table-0. (see appendix). The graph of the figure so generated is presented in graph-0 (see appendix).

A summary of the estimation experiment (table-0.0) suggests that the estimated parameters exhibit a remarkable consistency. This consistency is not observed when the figure in question is a circle. The possible reason for non-uniqueness of estimated parameters in case of a circle is that the circle is a degenerate case of innumerable many types of figures.

Table-0.0: A Summary of Estimation Experiments of the Gielis Parameters of the figure generated by the parameters in Table-0 (Appendix)							
Parameters (relating to)	True Values	Estimated Gielis Parameters of the Circle					
		Trial-1	Trial-2	Trial-3	Trial-4	Trial-5	
Shift in Center	C_x	0.0	-0.000098	0.001198	-0.000026	0.002430	0.000181
	C_y	0.0	0.003644	-0.001443	-0.000387	0.001956	0.000400
Length	a	1.0	0.634936	1.027326	1.097270	0.809647	1.098900
Width	b	1.0	1.910894	1.079770	1.198800	0.922626	1.098900
Exponent (Shape Parameters)	n_1	-2.0	-2.532928	-1.980109	-2.045747	-1.950055	-2.013009
	n_2	0.2	0.267146	0.233841	0.222509	0.224476	0.209174
	n_3	-0.3	-0.359563	-0.281780	-0.297176	-0.301441	-0.297706
Sub-Dimension (Vertices)	m	6.0	5.999738	6.001158	6.000014	5.996696	5.999843
Distance	s^2	0.0	0.002471	0.000333	0.000162	0.029486	0.000054

Estimation of the Parameters of a Rose Leaf: We have taken a sample rose leaf and 21 sample points on the trace of the same (see table-6.0 and graph-6.0). We have estimated the Gielis parameters by constrained Nelder-Mead algorithm². The algorithm was programmed in FORTRAN. Having estimated the parameters with four trial limits on parameters, we have generated 100 random points with those parameters and plotted them. Thus we have 4 sets of estimated parameters and the graphs generated from them, presented in tables 6.1 through 6.4 and graphs 6.1 through 6.4.

Two Best Estimated Parameters of the Sample Rose Leaf									
Parameters	c_x	c_y	a	b	n_1	n_2	n_3	m	s^2
Trial-4	0.946	-0.071	8.476	3.855	2.510	1.069	3.258	2.008	0.733
Trial-3	1.000	-0.105	2.000	2.000	0.831	1.093	2.093	2.012	0.867

Trials 3 and 4 (table 6.3 and 6.4) give best results in terms of the values of s^2 . These estimates are reproduced in the table above. Once again, we find that the estimated parameters make a large trade-off among themselves. We get a similar impression from table 7 and graph-7 as well.

Need of a Computer-based Interactive Software to Fit a Gielis Curve: It is highly unlikely that the available optimization methods would be able to best estimate the Gielis parameters from the empirical data. An interactive computer software is needed. I visualize software that divides the screen into two parts. On the left part we have the images of the shape of an object (call it IM_1 that will remain constant) together with the shape generated by the choice of different values of Gielis parameters - a, b, m, n_1, n_2 and n_3 (call it IM_2 that would change according to the selected values of these parameters). On the right we have six bars (each one with zero in the middle) representing values of the parameters and the meter reading counter for each of them. The user may select a particular value of any parameter by squeezing or stretching the bar through dragging the mouse. The effect of such a choice would be reflected in the left part of the screen, which would change the shape of the generated image IM_2 . A choice also may be there to choose a point on IM_2 and drag it (through mouse) closer to a point on IM_1 , which would work as a spline adjustment and this would be reflected in the values of the parameters on the right side. (This provision would possibly be harder to execute). Thus, by dragging the mouse, adjusting the bars (and the points on IM_2) the user would bring IM_2 as close to IM_1 as desired or possible. On the right side, the bars would read the values of the parameters.

A greater part of this has already been done. There are many examples of software that generate IM_2 for chosen Gielis parameters. The objective of this class of software is mostly designing or computer graphics, but not estimation of parameters for scientific projects. Now the need is to incorporate IM_1 and the choice of parameters not by selecting numbers but by dragging the mouse and reading them on their counters. I wish some software expert to do it.

Concluding Remarks: Although it would be too hasty to conclude that the best unique estimates of Gielis parameters for a given data cannot be obtained, our investigation has given us enough evidence to hold the opinion that we should take every estimate of these parameters rather cautiously. Estimation with several ranges on the parameters may provide rich information, but may not help to choose the best one among them. The investigation on the meaning of Gielis parameters would be greatly facilitated if suitable software were designed to estimate them quickly and accurately.

Notes:

1. Often we choose $p = 2$, which gives us the Euclidean distance, although any other measure of distance may be chosen. Two other measures of distance are often used: when $p=1$ (absolute distance), and when $p \rightarrow \infty$ (Chebyshev distance). All of them are particular cases of the Minkowski distance.
2. Kuester and Mize (1973) give the FORTRAN codes of unconstrained Nelder-Mead algorithm. There are some fatal (but simple) bugs in the program and therefore the program had to be modified. A provision to check for the constraints on the parameters was also added by the present author. All estimations in this paper are obtained by using the Nelder-Mead algorithm. An alternative and perhaps better method would be the Genetic Algorithm (see Mishra, 2006).

References

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Author's Contact: mishrasknehu@hotmail.com Ph: 913642550102 (R), 913642723175 (O)

Appendix

Table-0 : Parameters of the Figure (Generated) for Estimating Gielis Parameters [Random Number Seed = 1111]								
Parameters	c_x	c_y	a	b	n_1	n_2	n_3	m
Values	0.0	0.0	1.0	1.0	-2.0	0.2	-0.3	6.0

Table 1. Estimated Parameters of the Figure Generated by Parameters in Table 0.

```

-----
Estimated Parameters
-9.82467172E-005  0.00364419818  0.634936213  1.91089463  -2.53292799
 0.267145663  -0.359562755  5.99973774
Estimated data stored in file : GEST
-----
Minimum sum of squares of deviations at iteration= 98 is  0.00247050403
-----
Random Seed = 5531
Limits on Parameters
-0.10000  -0.10000  0.00000  0.00000  -4.00000  0.00000  -2.00000  5.00000
 0.10000  0.10000  2.00000  2.00000  0.00000  2.00000  0.00000  7.00000
ESTIMATED PARAMETERS
-0.00010  0.00364  0.63494  1.91089  -2.53293  0.26715  -0.35956  5.99974
End

```

Table 2. Estimated Parameters of the Figure Generated by Parameters in Table 0.

```

-----
Estimated Parameters
 0.00119847956  -0.00144315115  1.02732587  1.07977009  -1.98010921  0.23384133
-0.281700303  6.00115776
Estimated data stored in file : GEST
-----
Minimum sum of squares of deviations at iteration= 48 is  0.00333164632
-----
Random Seed = 6671
Limits on Parameters
-0.10000  -0.10000  0.00000  0.00000  -4.00000  0.00000  -2.00000  5.00000
 0.10000  0.10000  1.20000  1.10000  -1.00000  2.00000  0.00000  7.00000
ESTIMATED PARAMETERS
 0.00120  -0.00144  1.02733  1.07977  -1.98011  0.23384  -0.28170  6.00116
End

```

Table 3. Estimated Parameters of the Figure Generated by Parameters in Table 0.

```

-----
Estimated Parameters
-2.55999112E-005  -0.000307151289  1.09727037  1.19800009  -2.0457468
 0.222500579  -0.297176182  6.00001431
Estimated data stored in file : GEST
-----
Minimum sum of squares of deviations at iteration= 12 is  0.000162013239
-----
Random Seed = 7708
Limits on Parameters
-0.10000  -0.10000  0.90000  0.90000  -4.00000  0.00000  -2.00000  5.00000
 0.10000  0.10000  1.10000  1.20000  -1.00000  2.00000  0.00000  7.00000
ESTIMATED PARAMETERS
-0.00003  -0.00039  1.09727  1.19800  -2.04575  0.22251  -0.29718  6.00001
End

```

Table-4. Estimated Parameters of the Figure Generated by Parameters in Table-0.

```

Estimated Parameters
0.00243017473 0.00195562909 0.009647381 0.922626078 -1.95005502
0.224476472 -0.301441073 5.99669552
Estimated data stored in file : GEST

-----
Minimum sum of squares of deviations at iteration= 84 is 0.0294862464

-----
Random Seed = 2227
Limits on Parameters
-0.10000 -0.20000 0.80000 0.80000 -4.00000 0.00000 -2.00000 5.00000
0.20000 0.10000 1.20000 1.10000 -1.00000 3.00000 0.00000 8.00000
ESTIMATED PARAMETERS
0.00243 0.00196 0.00965 0.92263 -1.95006 0.22448 -0.30144 5.99670
End

```

Table-5. Estimated Parameters of the Figure Generated by Parameters in Table-0.

```

Estimated Parameters
0.000180780597 0.000400373188 1.09889984 1.09889984 -2.01300907
0.209174335 -0.297706425 5.99984264
Estimated data stored in file : GEST

-----
Minimum sum of squares of deviations at iteration= 29 is 5.37218548E-005

-----
Random Seed = 4773
Limits on Parameters
-0.10000 -0.10000 0.90000 0.90000 -3.00000 0.00000 -1.00000 4.00000
0.10000 0.10000 1.10000 1.10000 -1.00000 1.00000 0.00000 8.00000
ESTIMATED PARAMETERS
0.00018 0.00040 1.09890 1.09890 -2.01301 0.20917 -0.29771 5.99984
End

```

Table-6.0: Sample points on the Trace of a Rose Leaf

x	y	x	y	x	y
-5.0	0.0	3.1	-1.9	-2.1	2.1
-4.3	-0.3	3.5	-1.3	3.4	1.0
-2.9	-1.5	3.5	-0.5	-3.0	1.5
-1.5	-2.4	3.1	0.0	-4.1	0.5
0.4	-2.9	2.9	1.9	-4.5	0.2
1.2	-2.8	-2.1	2.1	3.2	-0.2
2.4	-2.4	0.9	2.7	3.0	0.0

Table-7 : Parameters of the 10-petal Rose (Generated) for Estimating Gielis Parameters
[Random Number Seeds for true = 1111 and for estimated = 2211]

Parameters	c_x	c_y	a	b	n_1	n_2	n_3	m
True	0	0	1	1	2	-3	-2	10
Estimated	-0.000301	0.000166	1.286895	1.494436	1.299214	-0.999	-0.999	9.969419

Table-6.1: Estimated G_{est} Parameter of the Rose Leaf

```

Estimated Parameters
0.999877737 -0.186571817 4.98298414 2.98584755 2.32211852 1.24817646
3.77248539 2.82618993
Estimated data stored in file : GEST
-----
Minimum sum of squares of deviations at iteration= 62 is 0.985896812
Random Seed = 4321
Limits on Parameters
-1.000 -0.200 4.000 2.000 1.000 1.000 1.000 1.000
1.000 0.200 5.000 4.000 4.000 4.000 4.000 7.000
End

```

Table-6.2: Estimated G_{est} Parameter of the Rose Leaf

```

Estimated Parameters
0.874718487 -0.315421522 4.98628187 3.597543 2.76887356 1.48823687
3.5955832 2.86975389
Estimated data stored in file : GEST
-----
Minimum sum of squares of deviations at iteration= 35 is 1.69574177
Random Seed = 4321
Limits on Parameters
-1.000 -1.000 0.000 0.000 1.000 1.000 1.000 1.000
1.000 1.000 5.000 4.000 4.000 4.000 4.000 5.000
End

```

Table-6.3: Estimated G_{est} Parameter of the Rose Leaf

```

Estimated Parameters
1. -0.18456226 2. 0.831833489 1.89278786 2.89283384 2.81159644
Estimated data stored in file : GEST
-----
Minimum sum of squares of deviations at iteration= 73 is 0.866811693
Random Seed = 5417
Limits on Parameters
1.000 -1.000 0.000 0.000 0.000 0.000 0.000 1.000
1.000 1.000 2.000 2.000 3.000 3.000 3.000 9.000
End

```

Table-6.4: Estimated G_{est} Parameter of the Rose Leaf

```

Estimated Parameters
0.945911487 -0.8788251894 0.47597584 3.8549161 2.51884958 1.86939865
3.25887571 2.88765944
Estimated data stored in file : GEST
-----
Minimum sum of squares of deviations at iteration= 47 is 0.733218742
Random Seed = 5511
Limits on Parameters
-1.000 -1.000 1.000 1.000 1.000 1.000 1.000 1.000
1.000 1.000 9.000 6.000 20.000 20.000 20.000 20.000
End

```

```

Estimated Parameters
-0.000300606509 0.000165657912 1.28689456 1.49443603 1.29921412
-0.999000013 -0.999000013 9.96941948
Estimated data stored in file : GEST
-----
Minimum sum of squares of deviations at iteration= 86 is 1.30261636
Random Seed = 2211
Limits on Parameters
-1.000 -1.000 0.500 0.500 1.000 -5.000 -3.000 5.000
1.000 1.000 1.500 1.900 3.000 -1.000 -1.000 15.000
End

```

